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A Review of Moving Plane Method for Traversing Dimensions: Applications in Astrophysics

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Abstract

In this paper, we study applications of the moving plane method, and a brief review of this method is conducted. In astrophysical investigation, the Moving Plane Method (MPM) has proven to be an effective method for navigating the multidimensional landscapes of space. We give a summary of the MPM, its theoretical underpinnings, and its uses in astrophysics in this review paper. We demonstrate how the MPM has transformed our knowledge of celestial events, from the dynamics of galaxy structures to the large-scale evolution of cosmic structures, using a variety of examples. We also talk about the MPM's future possibilities and existing problems in improving astrophysical research.

Keywords: The moving plane method, symmetric solutions, boundary value problems

INTRODUCTION

Problems related to nonlinear Elliptic, Parabolic, and Hyperbolic equations can be experienced in many mathematical models of applied science such as chemical reactions, nuclear science, population dynamics, heat transfer and fluid dynamics, biological sciences, acoustics, astrophysics, geophysics, electrodynamics, and many other disciplines. In recent years, there have been various results about uniqueness, existence, critical exponent, global existence, blowing-up, and other properties of the solution; many researchers have discovered various maximum principles and their applications. The moving plane method is the most popular method among researchers.

MOVING PLANE METHOD

The moving plane method was invented by the Soviet mathematician Alexandrov [1] in 1958 and introduced in his study of surfaces of constant mean curvature. It is an easy way of using the maximum principles to obtain the qualitative properties of positive solutions of Elliptic, Parabolic, and Hyperbolic equations such as the monotonicity and symmetry for solutions of differential equations. It can also be used to obtain a priori estimates, to derive useful inequalities, and to prove non-existence of solutions. The moving plane method greatly enhances the power of maximum principles. Roughly speaking, the moving plane method is a continuous way of repeated applications of maximum principles [2].

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In the study of astrophysics, the size and complexity of the cosmos are ongoing challenges. Novel approaches that may efficiently traverse the multidimensional realm of astrophysical phenomena are needed to comprehend the complex dynamics of celestial objects and systems. The Moving Plane Method (MPM) is one such technique that has become increasingly popular in recent years.

OUTLINE OF MOVING PLANE METHOD

The partial differential model (MPM) was initially created as a mathematical tool for understanding partial differential equations.

Because of its capacity to traverse dimensions and capture the dynamics of complicated systems, the MPM has found extensive uses in astronomy. The MPM offers a dynamic viewpoint, in contrast to conventional techniques that rely on static data, enabling researchers to examine how astrophysical phenomena change over time.

We apply the Moving Plane Method as follows:

Let the Euclidean space $R^n$and u(x) be a positive solution of some partial differential equation in Euclidean space R^n. If we want to prove u(x) is monotone and symmetric in the given directions then notify that direction as X_1-axis.

Let $T_\lambda = \{x = (x_1, x_2, x_3, \ldots\ldots, x_n); x_1 = \lambda\}$ for any real number λ. This is the plane that we will move with the X_1-axis and perpendicular to the X_1-axis. Now define Σ_λ the region of the left to the plane in Ω i.e. $\Sigma_\lambda = \{x: x_1 < \lambda, x \in \Omega\}$, about the plane T_λ denoted by x^λ. So $x^\lambda = (2\lambda - x_1, x_2, x_3, \ldots., x_n)$. Now we compare the values of u(x) at two points x and x^λ. First, we show that u(x) is symmetric about the plane T_λ. For this, we have to prove that,

$$u(x) = u(x^\lambda)$$

To prove this, Let

$$\omega_\lambda(x) = u(x) - u(x^\lambda)$$

Now suppose that there exists some λ_0 such that $\omega_{\lambda_0}(x) = 0$ for all $x \in \Sigma_{\lambda_0}$. Usually, it can be done by using the following two steps.

Step-I: First we prove that for λ sufficiently negative we have,$\omega_\lambda(x) \geq 0$ for all $x \in \Sigma_\lambda$.Then we are efficient to start from this neighbourhood of $x_1 = -\infty$, next move the plane T_λ along the x_1 - direction to the right since $\omega_\lambda(x) \geq 0$ holds for all $x \in \Sigma_\lambda$.

Step-II: This way we move the plane continuously up to its limiting position. Now we, define $\lambda_0 = \sup\{\omega_\lambda(x) \geq 0 \text{ for all } x \in \Sigma_\lambda\}$.We prove that $\omega_{\lambda_0}(x) = 0$ for all $x \in \Sigma_{\lambda_0}$ to show that u is symmetric about plane T_{λ_0}. We get a contradiction argument of method. We prove that if $\omega_{\lambda_0}(x) \neq 0$, then there exists $\lambda \geq \lambda_0$ such that $\omega_{\lambda_0}(x) \geq 0$. We get a contradiction to the definition of λ_0.

From the above showing, it is clear that the key to the moving plane method is to form an inequality $\omega_\lambda(x) \geq 0$ holds for all $x \in \Sigma_\lambda$,

Applications of the Moving Plane Method: Galactic Dynamics

The investigation of galactic dynamics is one of the main astrophysics' implications of the MPM. Researchers can examine the development and evolution of galactic structures by using the MPM to mimic the motion of stars and gas within galaxies. This involves comprehending how dark matter, star feedback, and gravitational interactions affect the morphology and kinematics of galaxies.

Cosmic Filaments and Large-Scale Structure

Research into the cosmic web, or the large-scale structure of the universe made up of interconnecting filaments of galaxies, has also benefited greatly from the MPM. Scientists can use the MPM to understand the development and transformation of cosmic structures as well as shed illumination on the fundamental mechanisms that drive cosmic growth and clustering by tracking the migration of matter along galactic threads.

Galaxy Development and Evolution

The study of galaxy development and evolution is an additional field in which the MPM shines. Through the MPM, scientists can simulate the gravitational collapse of primordial gas clouds and the ensuing galaxy formation, allowing them to investigate a range of galaxy formation scenarios such as hierarchical merging, gas accretion, and feedback processes. This makes it possible to have a thorough understanding of the wide variety of galaxies that have been seen in the universe.

Simulations of the Universe

Within the field of cosmology, the MPM is essential for modelling the universe's evolution at cosmic length scales. Cosmological computations predicated on the MPM enable insights into cosmic phenomena that include background radiation from cosmic microwave waves, large-scale structure development, and galaxy distribution across cosmic time by combining the dynamics of dark matter, baryonic matter, and dark energy.

Gravitational Lensing

The curvature of light by huge objects, or gravitational lensing, provides a special window into the universe's matter distribution. Researchers can map the distribution of dark matter, limit cosmological parameters, and investigate the nature of dark energy by using the MPM to model gravitational lensing events. This makes accurate measurements of cosmic shear research, galaxy-galaxy lensing, and mass distributions in galaxy clusters possible.

In this paper, we review of use of the moving plane method in elliptic boundary value problems.

In 1958, the Moving plane method was presented by A.D. Alexandrov in [1] known as the soap bubble Theorem, which states that Soap bubbles form a sphere shape instead of other shapes to minimize the surface area. The Author proved the following theorem.

Theorem 3.1[1]: Let $\Omega \in R^n$ be a bounded connected domain with boundary $S = \partial\Omega$ of class C^2. Then the mean curvature H of S is constant if and only if S is a sphere. A few years later, overdetermined problems of partial differential equations are solved by Serrin [3]. For that, he introduced the moving plane method and the proved following theorem;

Theorem 3.2[3]: Let Ω be a domain whose boundary is of class C^2. Suppose there exists a function $u \in C_2(\bar{\Omega})$ satisfying condition,

1. Suppose there exist a function $u = u(x) = u(x_1, x_2, x_3, \ldots\ldots, x_n)$ satisfying the Poisson differential equation
   $\Delta u = -1$ in Ω
2. Together with the boundary conditions
   $u = 0, \frac{\partial u}{\partial n} =$ constant on $\partial\Omega$

Then Ω is a ball and u have the specific form noted above. Serrin proved this theorem also for more general elliptic equations in the following theorem,

In 1991, Berestycki and L. Nirenberg [4] proved Monotone properties or symmetry solutions of nonlinear elliptic equations by using the Method of Moving Planes. Authors apply these methods for nonlinear elliptic equations of the type $F(x, u, Du, D^2u) = 0$. and obtained the following result

Theorem 3.3[4]: In the ball Ω: $|x| < R$ in R^n. Let u be a positive solution belonging to $C_2(\bar{\Omega})$ of $\Delta u + f(u) = 0$ *with* $u = 0$ *on* $\partial\Omega$. *If* $f \in C^1$, Then u is radially symmetric and the radial derivatives satisfy $u_r < 0$ for $0 < r < R$. and

If Ω is of class C^2 which is convex in the x_1 - direction and symmetric with respect to the $x_1 = 0$, then u is symmetric with respect to x_1 and $u_{x_1} < 0$ for $0 < x_1$ in Ω. Also, if Ω is an arbitrary bounded domain *in* R^n *which is convex in the* x_1 - direction and symmetric with respect to the $x_1 =$

0, and u is a positive solution belonging to $W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})$. Assume that u is symmetric with respect to x_1, and $u_{x_1} < 0$ for $0 < x_1$ in Ω.

In 1992[5], E.N. Dancer obtained results on the symmetry of positive solutions of boundary value problems for nonlinear elliptic equations by the moving plane method and this method has also been applied to some problems on half space.

The author showed symmetry and monotonicity properties of positive solutions of

$$-\Delta u = f(u)\ in\ \Omega$$

$$u = 0\ on\ \partial\Omega$$

(Note that: $u \in W^{1,2}(\Omega)$ and$f(u) \in L^{\infty}(\Omega)$)

For this author assumed that $f: R \to R$ is a locally Lipschitz function and Ω is a bounded domain in R^m. If $\lambda \epsilon R$, let $\Omega_\lambda = \{x \epsilon \Omega: x_1 > \lambda\}$ (where $(x_1, \ldots\ldots, x_m)\epsilon R^m$) and let P_λ be the reflection in the hyperplane $x = \lambda$ and proved theorem;

Theorem 3.4[5]: Assume that the above conditions hold, that $u(x) > 0$ in Ω and that $P_\lambda(\Omega_\lambda) \subseteq \Omega$ for $\lambda > \lambda^*$where T_{λ^*} intersects Ω.Then $u(P_{\lambda^*}x) \geq u(x)$ and $\frac{\partial u(x)}{\partial x_1} < 0\ if\ x \in \Omega_{\lambda^*}$.

Next, the author proved the theorem for a half-space case for this he considered the problem,

$$-\Delta u = f(u) \text{ in } T \tag{3.1}$$

$$u = 0 \text{ on } \partial T$$

Are increasing in x_1. Here $T = \{x \in R^m: x_1 > 0\}$.

Theorem 3.5[5]: Assume that f is C^1, with $f(0) > 0$ or both $f(0) = 0$ and $f'(0) = 0$ and that $\bar{u}$ is a non-trivial, non-negative bounded solution of above problem (3.1)on T with $\bar{u} = 0$ on ∂T.Then $\frac{\partial \bar{u}}{\partial x_1} > 0$ if $x_1 \geq 0$.

(Note: Half-space results are frequently required in “blowing up” arguments.)

In 1999 [6] Lucio Damascelli used the moving plane method to get the radial symmetry about a point $x_0 \in R^N$of the positive ground state solutions of the equation $-div\ (|Du|^{p-2}\ Du) = f(u)$ in R^N, in the case $1 < p < 2$ by assuming f to be locally Lipschitz continuous in $(0, +\infty)$ and nonincreasing near zero. To apply the Moving plane Method, they first proved a weak comparison theorem for solutions of differential inequalities in unbounded domains in R^N. The author proved the following theorem:

Theorem 3.6[6]: Under the assumptions,
1. f is locally Lipschitz continuous on $(0, \infty)$,
2. there exists $s_0 > 0$ such that f is nonincreasing on $(0, s_0)$.

If $u \in C^1(R^N) \cap W^{1,p}(R^N)$ is a solution of

$$-\Delta_p u(x) = f\big(u(x)\big)\ in\ R^N, u > 0,$$

$$u(x) \to 0\ as\ |x| \to \infty$$

For $1 < p < 2$ with $\Delta_p u = div\ (|Du|^{p-2}\ Du)$.

Then u is radially symmetric about some point $x_0 \in R^N$, i.e.$u = u(r)$, with

$r = |x - x_0|$ and $u'(r) < 0$ for all $r > 0$.

In 2015, D.B. Dhaigude and D.P. Patil [7] studied the symmetry properties of positive solutions of nonlinear elliptic boundary value problem of the type

$$\Delta u + f(|x|, u, \nabla u) = 0 \text{ in } R^n. \quad (3.2)$$

$$u(x) \to 0 \; as \; |x| \to \infty$$

For this, they used the moving plane method based on the maximum principle on unbounded domains and proved the following theorem,

Theorem [3.7][7]: Let $u \in C^2(R^n)$ be a positive solution of (3.2) with the following conditions

1. f is continuous in all of its variables and Lipschitz in u
2. $f(|x|, u, (p_1, p_2, p_3, \dots, p_{i-1}, -p_i, \dots., p_n)) = f(|x|, u, (p_1, p_2, p_3, \dots., p_n))$ *for all* $1 \le i \le n$
3. f *is nonincreasing in* $|x| = r > 0$, *for each fixed* $u \ge 0$.

Define u and ϕ as

$$u(r) = Sup\{u(x): |x| \ge r\}$$

$$\phi(r) = Sup\left\{\frac{\partial f}{\partial u}(|x|, u(x), \nabla u(x)): 0 \le u(x) \le u(r)\right\}$$

respectively, assume that there exists a positive function ω on $|x| \ge R_0$ for some $R_0 > 0$ satisfying

$$\Delta\omega + \phi(|x|) \le 0 \text{ in } |x| > R_0,$$

$$\lim_{|x| \to \infty} \frac{u(|x|)}{\omega(x)} = 0,$$

Then u must be radially symmetric about some point $x_0 \epsilon R^n$ and $u_r < 0$ for $r > 0$.

In 2016, D.P. Patil [8] used the maximum principle to the study symmetry properties of solutions for a system of semilinear elliptic boundary value problems with Neumann condition, to the elliptic operators more general than the Laplacian operator on the unit ball in n-dimensional Euclidian space with $n \ge 3$. and proved the following theorem,

Theorem [3.8][8]: Let $(u, v) \in C^2\,(B) \cup C^0\,(B)$ be a positive solution of the elliptic boundary value problem with Neumann condition,

$$a\,\Delta u + b_i \sum_{i=1}^{n} u_i = f_1(v)$$

$$a\,\Delta \mathrm{v} + \mathrm{b} \sum_{i=1}^{n} v_i = f_2(u)$$

$$\frac{\partial u}{\partial \eta} = g_1(v);$$

$$\frac{\partial v}{\partial \eta} = g_2(u);$$

Where $a: B \longrightarrow R$ is a bounded function and symmetric with respect to origin such that $a(x) > 0$ for all $x \in B$ and $b: B \longrightarrow R$ is a bounded and odd function. η is the normal vector to ∂B and $f_1, f_2, g_1, g_2 \in C(R)$. Further more f_1, f_2 are strictly increasing and g_1, g_2 are strictly decreasing.

Then the solution (u, v) must be radially symmetric with respect to the origin.

In 2018, D.P. Patil [9] obtained the symmetry properties of bi-harmonic equations of the type $\Delta^2 u + \alpha\, u = 0$ and $\Delta^2 u + f(u) = 0$ in $\Omega \subset R^2$,

The author used the method of moving planes, which is based on the Maximum principles in bounded domains and states, and proved the following theorem,

Theorem [3.9][9]: Let $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ be a non-constant solution of

$\Delta^2 u + \alpha u = 0 \; |\alpha| < 1, \alpha \epsilon R$ in $\Omega \subset R^n$

$\Delta u = 0$ on $\partial\Omega$

$u(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$

Define $U(r) = \sup\{(u): |x| \geq R\}$

$\Phi(r) = 1 \; B(r) = \{x \in R^n: |x| < r_0\}$

Assume that there exists a positive function w on $|x| \geq R_0$ for some $R_0 > 0$ satisfying,

$\Delta^2 u + \Phi(x) w \leq 0$ in $|x| > R_0$

$\Delta u = 0$ on $|x| > R_0$

$\lim_{|x| \to \infty} \frac{U(|x|)}{\omega(x)} = 0$

Then u must be radially symmetric about some point $x_0 \in R^n$ and $u_r \leq 0$ for $R_0 \geq 0$

Theorem [3.10][9]: Let

$\Delta^2 u + f(u) = 0$ in $\Omega \subset R^n$,

$\Delta u = 0$ on $\partial\Omega$

$u(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$

Assume that $f(u)$ is a positive nondecreasing and differentiable function.

Let $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$, Define $U(r) = \sup\{(u): |x| \geq R\}$

Then u must be radially symmetric about some point $x_0 \in \Omega$ and $u_r \leq 0$ for $r > 0$.

In 2019, D.P. Patil [Author] [10] studied the radial symmetry of positive solutions for semilinear elliptic boundary value problems in R^n by using the moving plane method. The author considered the problem of the form

$$\Delta u + b_1 \frac{\partial u}{\partial x_1} + b_2 \frac{\partial u}{\partial x_2} + \cdots + b_n \frac{\partial u}{\partial x_n} + f(|x|, u) = 0 \text{ in } R^n. \tag{3.3}$$

$$u(x) \to 0 \text{ as } |x| \to \infty \text{ when } n \geq 2 \tag{3.4}$$

State and proved the result [10]:

In the boundary value problem (3.3) & (3.4), assume that$(b_1, b_2, b_3, \ldots\ldots b_n) \in (0, \infty)$, $f(r, u)$ is non-increasing in $r > 0$ for each fixed $u \geq 0$.

Let $u \in C^2(R^n)$ be the solution of the boundary value problem (a), (b).

Define $U(r) = Sup\{u(x) || x| \geq r\}$

And $\Phi(r) = Sup \left\{ \frac{\partial f}{\partial u}(r, s) \middle| 0 \leq s \leq Sup\{u(x) || x| \geq r\} \right\}$.

Further, assume that there exists a positive function w on $|x| \geq R_0$ for some $R_0 > 0$ satisfying

$\Delta w + b_1 \frac{\partial w}{\partial x_1} + b_2 \frac{\partial w}{\partial x_2} + \cdots + b_n \frac{\partial w}{\partial x_n} + \Phi(|x|) w \leq 0$ in $|x| \geq R_0$

And $\lim_{|x|\to\infty} \frac{u(|x|)}{\omega(x)} = 0$

Then u must be radially symmetric about some point $x_0 \in R^n$ and $u_r < 0$ for $r > 0$.

In 2019 D. Berman [11] investigated the qualitative behavior of classical positive solutions to a large class of elliptic problems in symmetric domains of R^n. The qualitative approaches of the method of moving planes have been studied by establishing maximum and comparison principles and using these together to show that all positive classical solutions of $-\Delta u = f(u)$ in the unit ball and $u = 0$ on the boundary of the unit ball are radially symmetric about the origin, provided f satisfies a Lipschitz condition.

The author used the moving plane method for symmetry of solutions in 2019, by using only a handful of elementary concepts. He used Hopf's lemma for elliptic operators; strong and weak maximum principles for elliptic operators and various comparison principles to prove the following theorem.

Theorem 3.11[11]: Let $u \in C^2\ (B_1(0) \in C(\bar{B}_1(0))$ be a positive solution of

$-\Delta u = f(u)$ in $B_1(0)$

$u = 0$ on $\partial B_1(0)$

Then, u is radially symmetric and monotone decreasing about the origin.

Let $u \in C^2(R^n)$ be a positive solution of the equation $-\Delta u = u^p$ in R^n, with $n \geq 3, p := \frac{n+2}{n-2}$ and such that $u \in O(|x|^{2-n})$. Then there exists a point $x_0 \in R^n$ about which is u radially symmetric and monotone decreasing.

In 2022, D.P. Patil [12] used the moving plane method which is based on the maximum principle on unbounded domains, and obtained the symmetry result for the solutions of a system of non-linear elliptic boundary value problems.

The author considered the following equation and proved the theorem,

$$\Delta u(x) + g_1(|x|)e^{u(x)} + g_2((|x|)e^{v(x)} = 0$$

$$\Delta v(x) + g_3(|x|)e^{u(x)} + g_4(|x|)e^{v(x)} = 0 \text{ in } R^3 \tag{3.5}$$

$$u(x) \longrightarrow 0, v(x) \longrightarrow 0 \text{ as } |x| \longrightarrow \infty \tag{3.6}$$

Where $g_1(x), g_2(x), g_3(x), g_4(x)$ are locally held continuous in $(0, \infty)$, satisfying $u^+ \in L^\infty(R^3)$ and $v^+ \in L^\infty(R^3)$.

Theorem[3.12][12]: Let $v(x)$ satisfy differential inequality

$L(v) \geq 0$

in a domain. Where L is uniformly elliptic. If there exists a function $\omega(x) > 0$ on $\Omega \cup \partial\Omega$ and

$L(\omega) \leq 0$ in Ω.

Then $\frac{v(x)}{\omega(x)}$ can not attain a nonnegative maximum at a point P on $\partial\Omega$, which lies on the boundary of a ball in Ω, if $\frac{v}{w}$ is not constant then,

$\frac{\partial}{\partial\eta}\left(\frac{v}{\omega}\right) > 0$ at P

Where, $\frac{\partial}{\partial\eta}$ denotes the outward directional derivative.

Let ω be a harmonic function in R^3 and $\omega(x) = 0\ (|x|)$ as $|x| \longrightarrow \infty$.

Then ω must be constant.

Theorem[3.13][12]: Assume that g_1, g_2, g_3, g_4 satisfies

$$\lim_{|x|\to\infty} \sup r^{\alpha_i} g_i(r) < \infty \text{ for some } \alpha_i > 0, i = 1,2,3,4.$$

Let (u, v) be the solution of the system (3.5) satisfying $u^+ \epsilon\ L^\infty(R^3)$, where $u^+ = \max\{0, u\}, v^+ = \max\{0, v\}$ and

$$0 < \frac{1}{3\omega_3}\int_{R^3}(g_1(|x|)e^{u(x)} + g_2\left((|x|)e^{v(x)}\right)dx = \beta_1 < \infty$$

$$\text{And } 0 < \frac{1}{3\omega_3}\int_{R^3}(g_3(|x|)e^{u(x)} + g_4\left((|x|)e^{v(x)}\right)dx = \beta_2 < \infty$$

If $\alpha_i + \beta_j > 3$, then (u, v) must be radially symmetric.

CONCLUSION

Radial symmetry is a very important property when considering solutions to different equations and systems in R^n of elliptic boundary value problems. It can be found out by using moving plane method with the help of various maximum principles.

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